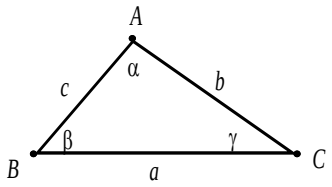
3.18 The Law of Cosines



The Law of Cosines: In any triangle,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

SSS: Use the fact that the largest angle is across from the longest side of the triangle to solve for the largest angle using the law of cosines. (For example, if *c* is the longest side, use the equation $c^2 = a^2 + b^2 - 2ab\cos\gamma$ to solve for γ.) Then use the law of sines to find the remaining angles, which will both be acute.

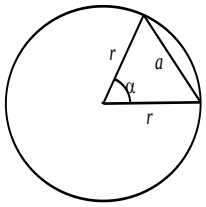
Example: $a = 3.8,\ b = 9.6,\ c = 7.7$

SAS: Find the length of the third side using the law of cosines. Use the law of sines to find the angle across from the shorter of the two given sides. Find the remaining angle by subtracting the first two from 180°.

Example: $b = 5.8,\ c = 3.6,\ \alpha = 39.5^\circ$

Example: Jan and Dean started hiking from the same location at the same time. Jan hiked at 4 mph with bearing N12°E, and Dean hiked at 5 mph with bearing N31°W. How far apart were they after 6 hours? Round to the nearest tenth of a mile.

Length of a Chord: If a chord of length a is intercepted by a central angle α in a circle of radius r, then $a = r\sqrt{2-2\cos\alpha}$. (This formula is derived from the law of cosines.)

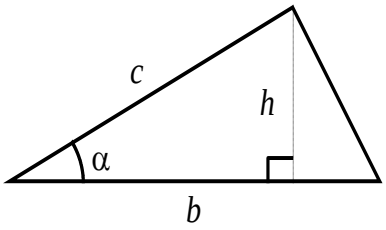


Example: Find the length of the chord intercepted by a central angle of 33.8° in a circle of radius 22.4 ft.

Area of a Triangle

The formula $A = \frac{1}{2}bh$ gives the area of a triangle.

In the diagram at the right, $\sin\alpha = \frac{h}{c}$, so $h = c\sin\alpha$. Using substitution, we derive the formula $A = \frac{1}{2}bc\sin\alpha$.



Depending on which angles and sides are known, the formulas $A = \frac{1}{2}ac\sin\beta$ and $A = \frac{1}{2}ab\sin\gamma$ can also be used.

Using the law of cosines, it is possible to derive a formula for the area of a triangle that involves only the lengths of the sides of the triangle. The formula is known as "Heron's Formula" after Heron of Alexandria, who is believed to have discovered it around AD 75.

Heron's Formula: The area of a triangle with sides of lengths a, b, and c is given by:

$$A = \sqrt{S(S-a)(S-b)(S-c)}, \text{ where } S = (a+b+c)/2.$$

Examples:

Find the area of the triangle with $\alpha = 39.4^\circ$, $b = 12.6$, and $c = 13.7$

Find the area of a triangle with $\alpha = 56.3^\circ$, $\beta = 41.2^\circ$, and $a = 9.8$

Find the area of the triangle with $a = 12$, $b = 8$, and $c = 6$

Find the area of a triangle with $a = 346$, $b = 234$, and $c = 422$